Local scale invariance as dynamical space-time symmetry in phase-ordering kinetics

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The scaling of the spatiotemporal response of coarsening systems is studied through simulations of the two-dimensional (2D) and 3D Ising model with Glauber dynamics. The scaling functions are consistent with the prediction of local scale invariance, thereby suggesting the extension of dynamical scaling to a space-time dynamical symmetry.

Ageing phenomena are observed in a broad variety of systems with slow relaxation dynamics. Useful insight may often be gained via the consideration of simple ferromagnetic models (rather than genuinely glassy systems), quenched to a temperature $T < T_c$ from a disordered initial state into the low-temperature phase. For their study two-time quantities, rather than the more usual one-time quantities, are particularly useful, see Refs. [1–3] for recent reviews. Those display dynamical self-similarity in the ageing regime where the order-parameter autocorrelator $C(t,s) = \langle \phi(t)\phi(s)\rangle$ decays from its plateau value $q_{EA} = M_{eq}^2$ to zero (M_{eq} is the spontaneous magnetization). In this regime both s and $\tau = t - s$ are much larger than the microscopic time scale (set to unity). Similarly, the autoresponse function, defined as $R(t,s) = \delta\langle\phi(t)\rangle/\delta h(s)$ with $t > s$ shows a scaling behavior such that the scaling laws

$$C(t,s) \simeq M_{eq}^2 f_C(t/s), \quad R(t,s) \simeq s^{-1-a} f_R(t/s) \qquad (1)$$

with $f_C(1) = 1$ are found to hold for a broad range of models [1–3]. For $x \gg 1$, the scaling functions usually fall off as $f_C(x) \sim x^{-\lambda_C/z}$ and $f_R(x) \sim x^{-\lambda_R/z}$, where z is the dynamic critical exponent, λ_C is the autocorrelation exponent [4,5], and λ_R is the autoresponse exponent [6]. We shall focus here on phase ordering in the Glauber-Ising model in $d > 1$ dimensions where $a = 1/z = 1/2$ is expected [2,7]. Recent arguments [8] leading to $a = 1/4$ in the two-dimensional (2D) Glauber-Ising model have been rejected through a detailed study of the scaling of $R(t,s)$, the results of which reconfirm $a = 1/2$ [9].

Equation (1) states that the two-time quantities evaluated at the *same* spatial location transform covariantly under a global rescaling of time $t \to bt$ with b constant. Recently, it has been proposed that the response functions should transform covariantly under *local* scale transformations with $b = b(t)$, but with time translations excluded [10]. By analogy with conformal invariance, in particular covariance of R under the so-called "special" transformations which transform time as $t \to t/(1 + \gamma t)$, is assumed. If that is the case, the exact autoresponse scaling function becomes [10,11]

$$f_R(x) = r_0 x^{1+a-\lambda_R/z}(x-1)^{-1-a} \qquad (2)$$

and depends on the universality classes only through the values of the exponents a and λ_R/z, while r_0 merely is a normalization constant. This prediction of local scale invariance (LSI) has been confirmed in a variety of physically very different models, see Ref. [10], and references therein. Still, better evidence in favor of LSI than mere phenomenology would be desirable.

The origin of LSI can be understood in the special case $z = 2$ (which is realized in all cases of phase-ordering kinetics we are going to consider [12]). A Ward identity can be written down such that if the system is known to be (i) scale invariant with $z = 2$ and (ii) Galilei invariant, then its invariance under the full group of local scale transformations follows [13]. Therefore, scale invariance and Galilei invariance appear as the building blocks for LSI. So far, no explicit test of Galilei invariance in phase-ordering systems was ever performed.

Tests of Galilei invariance require the study of the time *and* space dependence of response functions. For phase ordering, it is natural to expect a scaling behavior of the linear response function

$$R(t,s;\mathbf{r}) = \frac{\delta\langle\phi_{\mathbf{r}}(t)\rangle}{\delta h_{\mathbf{0}}(s)}\bigg|_{h=0} = s^{-1-a}F_R\left(\frac{t}{s}, \frac{\mathbf{r}}{(t-s)^{1/z}}\right)$$

with the scaling function $F_R(x,\mathbf{u})$ and $F_R(x,\mathbf{0}) = f_R(x)$. For $z = 2$, Galilei invariance (combined with the Ward identity of LSI) predicts [10,13,14]

$$R(t,s;\mathbf{r}) = R(t,s)\exp\left(-\frac{\mathcal{M}}{2}\frac{\mathbf{r}^2}{t-s}\right), \qquad (3)$$

where $R(t,s)$ is the autoresponse function given by Eqs. (1) and (2) and $\mathcal{M}$ is a direction-dependent nonuniversal constant. Equation (3) gives the full spatiotemporal scaling of the linear response. We shall successfully test it in the 2D and 3D kinetic Glauber-Ising models and provide thereby evidence of Galilei invariance in a phase-ordering system. Since the 2D/3D Glauber-Ising model cannot be reduced to a free-field theory, it is nontrivial that its *exact* response function should take the simple free-field form (3).

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As precise simulational data for the autoresponse $R(t,s)$ are difficult to obtain, it is convenient to study instead the thermoremanent magnetization M_{TRM} [15]

$$TM_{\mathrm{TRM}}(t,s)/h_{(0)}=\rho(t,s)=T\int_0^s du\, R(t,u). \qquad (4)$$

We consider the Ising model on a hypercubic lattice, with periodic boundary conditions and Hamiltonian $\mathcal{H}=-\Sigma_{(\mathbf{i},\mathbf{j})}\sigma_{\mathbf{i}}\sigma_{\mathbf{j}}$. We use heat-bath dynamics defined through the stochastic rule

$$\sigma_{\mathbf{i}}(t+1)=\pm 1 \ \text{ with probability } \ \frac{1}{2}\{1\pm\tanh[h_{\mathbf{i}}(t)/T]\} \qquad (5)$$

with the local field $h_{\mathbf{i}}(t)=\Sigma_{\mathbf{a}(\mathbf{i})}\sigma_{\mathbf{a}}(t)$ and where $\mathbf{a}(\mathbf{i})$ runs over the nearest neighbors of the site $\mathbf{i}$. $M_{\mathrm{TRM}}(t,s)$ is measured by applying a quenched spatially random magnetic field $\pm h_{(0)}$ for times between the quench at $t=0$ and the waiting time s [15]. The presence of this external field then changes the local field in Eq. (5) to $h_{\mathbf{i}}(t)=\Sigma_{\mathbf{a}(\mathbf{i})}\sigma_{\mathbf{a}}(t)\pm h_{(0)}$.

It has been understood recently that the scaling behavior of M_{TRM} for $s\gg 1$ is not a simple power law but rather shows a crossover behavior [8,16]. It is of the form [9]

$$\rho(t,s)=r_0 s^{-a} f_M(t/s)+r_1 s^{-\lambda_R/z} g_M(t/s) \qquad (6)$$

provided the system was initially prepared at infinite temperature. Here the correction term can be estimated as $g_M(x)\simeq x^{-\lambda_R/z}$, while the scaling function $f_M(x)$ can be found from local scale invariance using Eqs. (1), (2), and (4) and is given explicitly by Eq. (5.47) in Ref. [10]. Finally, $r_{0,1}$ are normalization constants. In practice, it turns out that λ_R is quite close to its lower bound $d/2$ and in particular in 2D case, the correction to the leading scaling behavior is sizeable. Before any meaningful study of the spatiotemporal response can be carried out, the correction term must be subtracted off.

As explained in Ref. [9], this can be carried out by fixing a value of $x=t/s$. Then $r_{0,1}$ are obtained by fitting Eq. (6) to the computed thermoremanent magnetization M_{TRM}. In order to illustrate the quality of this procedure, we show in Fig. 1 data for the scaling function $f_M(x)$ obtained after subtraction of the correction term $r_1 s^{-\lambda_R/z} g_M(t/s)$ from the integrated autoresponse ρ of the 2D and 3D Glauber-Ising model. Statistical error bars in 2D case are smaller than the symbol size and in 3D case are of the order of the scatter in the data. In Table I, we list the values of λ_R [4,17] and the constants $r_{0,1}$ which were determined at $x=7$. We find a nice data collapse and a clear agreement with the LSI prediction.

Having thus checked the expected scaling of the autoresponse function, we can now turn towards the spatiotemporal response. Again, we consider the integrated response rather than $R(t,s;\mathbf{r})$ because it is considerably less affected by noise. In order to fix the nonuniversal parameter $\mathcal{M}$ in Eq. (3), we form
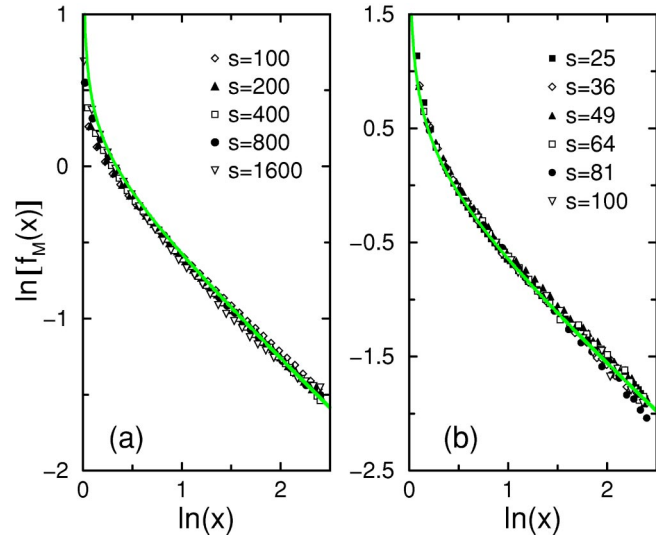

FIG. 1. Scaling of the integrated response function ρ for the Glauber-Ising model. (a) 2D at temperature $T=1.5$ and (b) 3D at $T=3$. The symbols correspond to different waiting times. The full curves are obtained by integrating Eq. (2).

$$\rho_0(t,s;\mathbf{r})=T\int_0^s d\tau\, R(t,s-\tau;\mathbf{r}) \qquad (7)$$

which is measured on the lattice by keeping a small random magnetic field $\pm h_{(0)}$ until the waiting time s at the site $\mathbf{0}$ and then observing the thermoremanent magnetization $M_{\mathrm{TRM}}(t,s;\mathbf{r})=h_{(0)}\rho_0(t,s;\mathbf{r})/T=\overline{\langle h_{\mathbf{0}}\,\sigma_{\mathbf{r}}(t)\rangle}/T$ at a different site $\mathbf{r}$ (where $\langle\cdots\rangle$ indicates an average over the thermal noise whereas the bar means an average over the random field distribution). We computed the spatially and temporally integrated response for the Ising model with Glauber dynamics in two and three dimensions. The two-dimensional systems usually contained 300^2 spins on a square lattice, whereas in three dimensions cubic lattices with typically 60^3 sites were considered. Some Monte Carlo simulations were also done for other system sizes in order to check against finite-size effects. Since $\rho_0(t,s;\mathbf{r})$ is very noisy, one must average over a large number of runs with different realizations of the thermal noise and of the spatially random magnetic field. For every waiting time we avaraged over at least 50 000 different runs. The whole study took about 2×10^5 CPU h on a SGI Origin 3800 parallel computer.

Following the same lines as in Ref. [9], we arrive at the scaling form [18], where $\mathbf{r}$ varies along a fixed direction on the lattice

TABLE I. Values of the autoresponse exponent λ_R and of the parameters r_0, r_1 and $\mathcal{M}$ in the Glauber-Ising model in two dimensions at $T=1.5$ and in three dimensions at $T=3$.

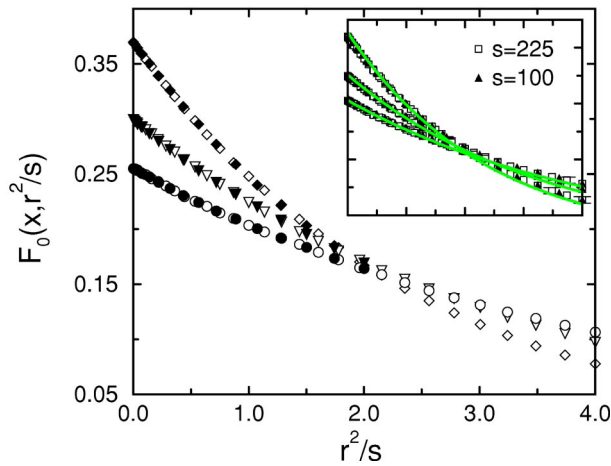|	λ_R	r_0	r_1	$\mathcal{M}$
2D	1.26	2.65 ± 0.05	-2.76 ± 0.05	4.08 ± 0.04
3D	1.60	0.31 ± 0.02	0.61 ± 0.02	4.22 ± 0.05

FIG. 2. Scaling of the temporally integrated response function ρ_0 for the 2D Glauber-Ising model at $T=1.5$ and $s=225$. Data obtained along the (10) [(11)] direction are shown by open (filled) symbols. Diamonds, $x=5$; triangles, $x=7$; and circles, $x=9$. Inset: Determination of the mass $\mathcal{M}$, see main text. Some typical error bars are shown in the inset.

$$\rho_0(t,s;\mathbf{r}) \simeq \rho_\infty(t-s;\mathbf{r}) + r_0 s^{-a} F_0\left(\frac{t}{s}, \mathcal{M}\frac{r^2}{s}\right)$$

$$+ r_1 s^{-\lambda_R/z} G_0\left(\frac{t}{s}, \mathcal{M}\frac{r^2}{s}\right), \qquad (8)$$

where the universal scaling functions read

$$F_0(x,y) = \int_0^1 dv \, \exp[-y/2(x-1+v)] h_R(x,v),$$

$$G_0(x,y) \simeq x^{-\lambda_R/z} e^{-y/2x}, \qquad (9)$$

with $h_R(x,v) := f_R[x/(1-v)](1-v)^{-1-a}$, where $\mathcal{M}$ is a direction-dependent new parameter. In this work we choose T such as to keep anisotropy effects small and also to avoid the crossover to critical dynamics at $T=T_c$. This is illustrated in Fig. 2, for the 2D case at $T=1.5$, where we plot F_0 as function of r^2/s. For the displayed data points r varies between 1 and $2\sqrt{s}$ when going along the (10) direction and between $\sqrt{2}$ and $\sqrt{2s}$ when going along the (11) direction. At the considered temperature any anisotropy is very small and the values of $\mathcal{M}$, determined separately in the two directions, coincide, $\mathcal{M}_{(10)} = \mathcal{M}_{(11)} = \mathcal{M}$. Similar results also hold in 3D at $T=3$. We fix $\mathcal{M}$ at $x=7$ (see Table I and inset of Fig. 2) using Eq. (8) and then obtain a parameter-free prediction for other values of x. This is shown in the inset of Fig. 2, where the LSI-prediction is compared to numerical data for two additional values of x and two different waiting times. Since by now all nonuniversal parameters are fixed, this constitutes by itself a quantitative test of local scale invariance.

A fuller test of the spatiotemporal response is obtained by considering the spatially and temporally integrated response function

$$\frac{d\rho(t,s;\mu)}{d\Omega} = T \int_0^s du \int_0^{\sqrt{\mu s}} dr \, r^{d-1} R(t,u;\mathbf{r}), \qquad (10)$$

where the space integral is along a straight line of length $\Lambda = \sqrt{\mu s}$ but we do allow for the possibility of an anisotropy as a function of the solid angle Ω. Such anisotropies are known to occur if $T<T_c$ [19]. As before, we derive from Eq. (3) the scaling form [18]

$$\frac{d\rho(t,s;\mu)}{d\Omega} = \rho^{(1)}(t-s,\mathbf{r}^2) + r_0 s^{d/2-a}\rho^{(2)}(t/s,\mu)$$

$$+ r_1 s^{d/2-\lambda_R/z}\rho^{(3)}(t/s,\mu) \qquad (11)$$

with the explicit scaling functions

$$\rho^{(2)}(x,\mu) = \frac{\mu^{d/2}}{d}\int_0^1 dv \, h_R(x,v)\mathcal{F}_d\left(\frac{\mathcal{M}\mu}{x-1+v}\right),$$

$$\rho^{(3)}(x,\mu) \simeq \frac{\mu^{d/2}}{d} x^{-\lambda_R/z}\mathcal{F}_d\left(\frac{\mathcal{M}\mu}{x}\right), \qquad (12)$$

$$\mathcal{F}_d(y) = e^{-y/2}\,_1F_1\left(1,1+\frac{d}{2};\frac{y}{2}\right).$$

This is the general expression for the scaling of the spatiotemporally integrated response function. If we fix μ and let $x=t/s$ vary, the form of the scaling function of $s^{a-d/2}d\rho/d\Omega$ merely depends on μ. We stress that the exponent λ_R and the free parameters $r_0, r_1, \mathcal{M}$ are now all fixed such that there remains no free fitting parameter at all when comparing Eq. (11) with our numerical data.

It is of interest to compare the maximal distance $r_{\max}$ accessed by our simulation with the physical length scale of the problem, for example, the domain size $L(s)$ at the instant the magnetic field is turned off. From the correlation function, in 2D we estimate $L(s) \approx 3,6,9.5$ for $s=25,100,225$, respectively, with $h_{(0)} = 0.05$. In 3D, we find $L(s) \approx 2.1,3.3,4$ for $s=25,64,100$. As expected $L(s) \sim s^{1/z}$, with $z \approx 2$. The value $\mu=4$ ($\mu=2$) corresponds to $r_{\max}/L(s) \approx 3.2$ (3.4) in 2D (3D) case, respectively and we see that the data probe the large-distance region, well beyond the domain size $L(s)$.

The integrated spatiotemporal response in 2D at $T=1.5$ is shown in Fig. 3 for two values of μ and where the correction $\rho^{(3)}$ was already subtracted off. We find a nice scaling behavior over the whole range of waiting times we could consider. We stress that the agreement between local scale invariance (3) and our data for several values of μ is a real test, since no free parameter remains. In particular, both the height and the position of the maximum of the scaling function for $\mu=2$ and 4 are completely fixed. We can conclude that the LSI prediction [Eqs. (11) and (12)] is fully vindicated. Here the exact functional form of the scaling function of the spatiotemporal response of a generic nonequilibrium spin system is found.
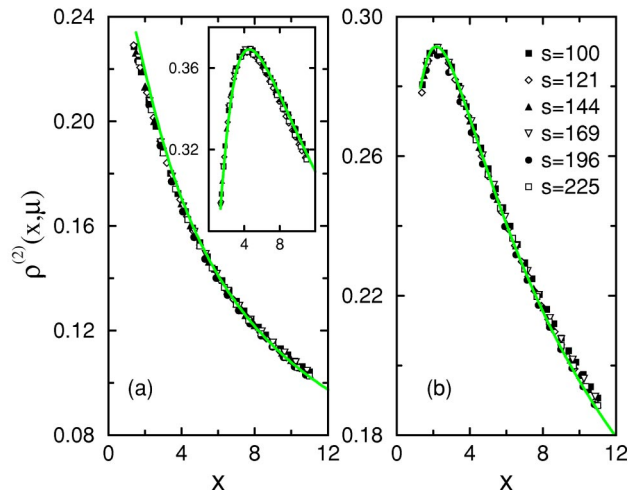
FIG. 3. Scaling of the integrated spatiotemporal response function ρ for the 2D Glauber-Ising model at $T = 1.5$ and at (a) $\mu = 1$ (inset: $\mu = 4$) and (b) $\mu = 2$ [integrating (i) along the (10) direction and (ii) along the (11) direction]. The full curves are from Eq. (12). Error bars are smaller than the symbol size.

FIG. 4. Scaling of the integrated spatiotemporal response function ρ for the 3D Glauber-Ising model at $T = 3$ and at (a) $\mu = 1$ and (b) $\mu = 2$ [integrating (i) along the (100) direction and (ii) along the (110) direction]. The full curves are from Eq. (12). Some typical error bars are also shown.

Similarly, the integrated spatiotemporal response in 3D case at $T = 3$ is shown in Fig. 4. Again, a nice scaling behavior in perfect agreement with LSI is found.

Prediction (3) is also verified in a few exactly solvable models with a nonconserved order parameter and undergoing phase-ordering kinetics with $z = 2$. In particular, Eq. (3) holds in the d-dimensional kinetic spherical model with $d > 2$ quenched to either $T < T_c$ or $T = T_c$ and independent of initial correlations [6,16,20–22].

Summarizing, we presented a quantitative study of the scaling of the spatio-temporal response in coarsening sys-
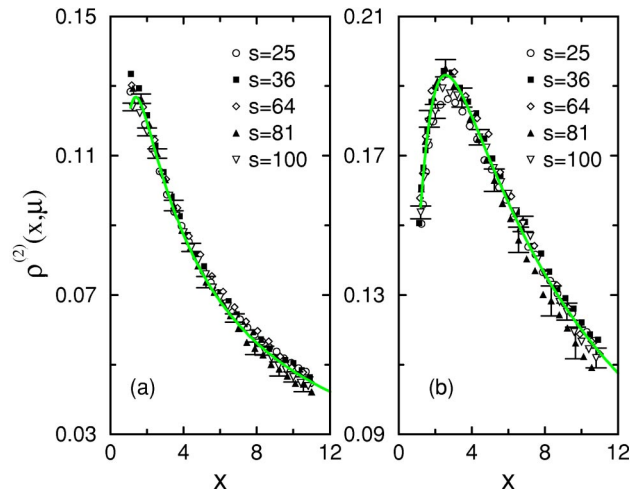
tems. We find evidence for Galilei invariance in the ageing regime of a phase-ordering system. Our result provides a strong indication that LSI is indeed a true *space-time* dynamical symmetry of statistical systems undergoing phase-ordering kinetics. The presence of this symmetry in a nonequilibrium phase transition was not anticipated.

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